

# Density perturbations in general modified gravitational theories

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(Dated: June 3, 2010)

We derive the equations of linear cosmological perturbations for the general Lagrangian density  $f(R, \phi, X)/2 + \mathcal{L}_c$ , where  $R$  is a Ricci scalar,  $\phi$  is a scalar field, and  $X = -\partial^\mu \phi \partial_\mu \phi/2$  is a field kinetic energy. We take into account a nonlinear self-interaction term  $\mathcal{L}_c = \xi(\phi) \square \phi (\partial^\mu \phi \partial_\mu \phi)$  recently studied in the context of “Galileon” cosmology, which keeps the field equations at second order. Taking into account a scalar-field mass explicitly, the equations of matter density perturbations and gravitational potentials are obtained under a quasi-static approximation on sub-horizon scales. We also derive conditions for the avoidance of ghosts and Laplacian instabilities associated with propagation speeds. Our analysis includes most of modified gravity models of dark energy proposed in literature and thus it is convenient to test the viability of such models from both theoretical and observational points of view.

## I. INTRODUCTION

The constantly accumulating observational data [1–3] have continuously confirmed that the Universe entered the phase of accelerated expansion after the matter-dominated epoch. This has motivated the idea that the gravitational law in General Relativity (GR) may be modified at cosmological distances to be responsible for the cosmic acceleration (see Refs. [4] for review). Many dark energy models based on the modification of gravity were proposed—including  $f(R)$  gravity [5] and generalizations [6], scalar-tensor theory [7], ghost condensation [8, 9], the Dvali-Gabadadze-Porati (DGP) braneworld model [10], and the Galileon model [11].

The modified gravity models of dark energy need to be constructed in such a way that Newton gravity is recovered at short distances for the consistency with solar-system experiments, while the deviation from GR can be allowed at large distances. On cosmological scales the effective mass of a scalar-field degree of freedom is required to be very small (of the order of the present Hubble constant) for realizing the cosmic acceleration today. Such a small mass generally leads to a long range fifth force incompatible with local gravity experiments [12], unless some mechanism screens the scalar-field interaction with standard matter.

There are two known mechanisms that allow a decoupling of the scalar field in local regions. The first is the so-called chameleon mechanism [13] in which the field mass is different depending on the matter density in the surrounding environment. If the field is sufficiently heavy in the regions of high density, a spherically symmetric body can have a “thin-shell” around its surface such that the effective coupling between the field and matter is suppressed outside the body. In  $f(R)$  gravity and scalar-tensor theory, the chameleon mechanism can be at work to avoid the propagation of the fifth force even if the bare scalar-field coupling with matter is of the order of unity [14–19]. In fact, a number of authors proposed viable dark energy models based on  $f(R)$  and scalar-tensor theories in the requirement that the field mass is sufficiently large in the regions of high density [16, 20–24] (see also Refs. [17, 25, 26]).

The second mechanism is the so-called Vainshtein mechanism [27] in which nonlinear effects of scalar-field self interactions lead to the recovery of GR at small distances. In massive gravity where a consistent massive graviton is determined by Pauli-Fierz theory [28], the linearization close to a point-like mass source breaks down inside the so-called Vainshtein radius  $r_*$ . One needs to take into account a nonlinear self-interaction for the radius smaller than  $r_*$ . In the DGP braneworld model [10] the Vainshtein mechanism is also at work through a field self-interaction of the form  $(r_c^2/m_{\text{pl}}) \square \phi (\partial^\mu \phi \partial_\mu \phi)$ , where  $\phi$  represents a brane-bending mode,  $r_c$  is a cross-over scale of the order of the Hubble radius  $H_0^{-1}$  today, and  $m_{\text{pl}}$  is the Planck mass [29–31]. Nonlinear effects lead to the decoupling of the field  $\phi$  from matter in the region where the energy density  $\rho$  is much larger than  $r_c^{-2} m_{\text{pl}}^2$ .

Although the Vainshtein effect screens the fifth force mediated by the brane-bending mode, the self-accelerating cosmological solution in the DGP model is plagued by the appearance of a ghost mode [31–33]. In order to realize self-accelerating solutions without ghosts, the equations of motion should be kept at second order in derivatives. This can be addressed by considering higher derivative scalar-field interactions that respect a constant gradient-shift in the Minkowski space-time, i.e.  $\partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu$  [34]. In addition to the term proportional to  $\square \phi (\partial^\mu \phi \partial_\mu \phi)$ , Nicolis *et al.* [34] derived other higher derivative interactions that keep the equations of motion at second order. This was extended to a more generalized covariant Galileon nonminimally coupled to the scalar curvature [35].

In the presence of a nonlinear self-interaction of the form  $\xi(\phi) \square \phi (\partial^\mu \phi \partial_\mu \phi)$ , where  $\xi(\phi)$  is a function of a scalar field

$\phi$ , Silva and Koyama [36] showed that it is possible to avoid the appearance of a ghost mode in the 4-dimensional Brans-Dicke theory [37]. Although such a term does not satisfy the Galilean invariance in the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological background, the equations of motion remain at second order. For the choice  $\xi(\phi) \propto \phi^{-2}$  there exists a de Sitter solution responsible for the late-time cosmic acceleration, while the Vainshtein mechanism allows to recover the General Relativistic behavior in the asymptotic past [36]. Recently this was extended to more general modified gravitational theories in which the late-time de Sitter solution is realized by the field kinetic energy [38].

In order to confront dark energy models with observations of large-scale structure, cosmic microwave background (CMB), and weak lensing, it is important to study the evolution of cosmological perturbations [39]. If the gravity is modified from GR, the effective gravitational coupling characterizing the growth rate of matter perturbations is subject to change. Hence, in general, modified gravity models leave distinct observational signatures compared to models based on GR. In this respect the evolution of matter perturbations has been extensively studied for a number of modified gravity models—including  $f(R)$  gravity [16, 20, 22, 40, 41], scalar-tensor theory [24, 42–46], the DGP model [47], and the Galileon model [36, 48–50] (see also Refs. [51]). In Ref. [41] the equation of matter perturbations was derived for the general Lagrangian  $f_1(R, \phi)/2 + f_2(\phi, X)/2$ , under the assumption that the mass of the field  $\phi$  is as light as the Hubble constant  $H_0$  today. This assumption can be justified for the field potential having a light mass during most of the cosmological epochs. In scalar-tensor models with a coupling of the order of unity between the field  $\phi$  and matter, the potential needs to be designed to have a large mass in the regions of high density so that the chameleon mechanism screens the fifth force [24]. Cosmologically the transition from the “massive” regime to the “massless” regime can occur during the matter era, depending on the wavelength of perturbations (including  $f(R)$  gravity) [20, 22, 24, 46]. Hence it is important to take into account such a mass term to estimate the epoch of transition and the resulting matter power spectrum.

In this paper we study the evolution of density perturbations for the general action (1) given below. In addition to the effect of the field mass, we discuss the effect of the nonlinear self-interaction  $\mathcal{L}_c = \xi(\phi)\Box\phi(\partial^\mu\phi\partial_\mu\phi)$  on the dynamics of perturbations. In other words we consider two effects for the recovery of GR in the past cosmological evolution—(i) the chameleon mechanism through the field mass  $M_\phi$ , and (ii) the Vainshtein mechanism through the self-interaction term  $\mathcal{L}_c$ . For the theories with  $f = f_1(R, \phi) + f_2(\phi, X)$  plus  $\mathcal{L}_c$ , we shall derive the equation of matter perturbations as well gravitational potentials under the quasi-static approximation [41, 42, 46, 52] on sub-horizon scales. Since our analysis is sufficiently general, this will be useful to place observational constraints on each model. We also derive conditions for the avoidance of ghosts and Laplacian instabilities, without recourse to any approximation. This is required for the construction of viable dark energy models free from theoretical inconsistency.

## II. PERTURBATION EQUATION IN GENERAL MODIFIED GRAVITATIONAL THEORIES

We start with the following 4-dimensional action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(R, \phi, X) + \xi(\phi) \Box\phi(\partial^\mu\phi\partial_\mu\phi) \right] + \int d^4x \mathcal{L}_m(g_{\mu\nu}, \Psi_m), \quad (1)$$

where  $g$  is the determinant of the metric  $g_{\mu\nu}$ ,  $f$  is a function in terms of the Ricci scalar  $R$ , a scalar field  $\phi$  and a kinetic term  $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ , and  $\xi(\phi)$  is a function of  $\phi$ .  $\mathcal{L}_m$  is a matter Lagrangian that depends on the metric  $g_{\mu\nu}$  and matter fields  $\Psi_m$ . If  $\xi(\phi)$  is constant, then this term respects the Galilean symmetry  $\partial_\mu\phi \rightarrow \partial_\mu\phi + b_\mu$  as well as the shift symmetry  $\phi \rightarrow \phi + c$  in the Minkowski space-time [34, 35, 48]. Here we consider a general function  $\xi(\phi)$  in terms of  $\phi$ , as this allows the possibility to give rise to the late-time cosmic acceleration without the appearance of a ghost [36, 49].

Varying the action (1) with respect to  $g_{\mu\nu}$  and  $\phi$ , we obtain the following field equations

$$FG_{\mu\nu} = \frac{1}{2}(f - RF)g_{\mu\nu} + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F + \frac{1}{2}f_{,X}\nabla_\mu\phi\nabla_\nu\phi + T_{\mu\nu}^{(c)} + T_{\mu\nu}^{(m)}, \quad (2)$$

$$\nabla_\mu(f_{,X}\nabla^\mu\phi) + f_{,\phi} = T^{(c)}, \quad (3)$$

where  $F \equiv \partial f/\partial R$ ,  $f_{,X} \equiv \partial f/\partial X$ ,  $f_{,\phi} \equiv \partial f/\partial\phi$ ,  $T_{\mu\nu}^{(m)} \equiv -(2/\sqrt{-g})\delta\mathcal{L}_m/\delta g^{\mu\nu}$  is the energy-momentum tensor of matter, and

$$T_{\mu\nu}^{(c)} = 2(\xi\partial^\lambda\phi\partial_\lambda\phi)_{,(\mu}\phi_{,\nu)} - 2\xi\Box\phi(\partial_\mu\phi\partial_\nu\phi) - \partial_\lambda(\xi\partial^\rho\phi\partial_\rho\phi)\partial^\lambda\phi g_{\mu\nu}, \quad (4)$$

$$T^{(c)} = -2[\xi_{,\phi}\Box\phi(\partial^\mu\phi\partial_\mu\phi) + \Box(\xi\partial^\mu\phi\partial_\mu\phi) - 2\nabla_\mu(\xi\Box\phi\partial^\mu\phi)]. \quad (5)$$

We consider the following perturbed metric about a spatially flat FLRW cosmological background with scalar metric perturbations  $\Psi$  and  $\Phi$  in a longitudinal gauge [53]:

$$ds^2 = -(1 + 2\Psi)dt^2 + a(t)^2(1 - 2\Phi)\delta_{ij}dx^i dx^j, \quad (6)$$

where  $a(t)$  is a scale factor with cosmic time  $t$ . We caution that our notations of  $\Psi$  and  $\Phi$  are opposite to those used in Ref. [41]. We decompose the field  $\phi$  into the background and inhomogeneous parts:  $\phi(t, \mathbf{x}) = \tilde{\phi}(t) + \delta\phi(t, \mathbf{x})$ . We consider a perfect fluid for the energy-momentum tensors  $T_{\mu\nu}^{(m)}$ , so that they can be decomposed as  $T_0^{0(m)} = -(\rho + \delta\rho)$ ,  $T_\alpha^{0(m)} = -(\rho + P)v_{,\alpha}$ , and  $T_\beta^\alpha(m) = (P + \delta P)\delta_\beta^\alpha$  (where  $v$  is a velocity potential). We also decompose  $T_{\mu\nu}^{(c)}$  and  $T^{(c)}$  into the background and perturbative parts, as  $T_{\mu\nu}^{(c)} = \tilde{T}_{\mu\nu}^{(c)} + \delta T_{\mu\nu}^{(c)}$  and  $T^{(c)} = \tilde{T}^{(c)} + \delta T^{(c)}$ . In the following, when we express background quantities, we drop the tilde for simplicity.

In the flat FLRW background we obtain the following equations

$$3FH^2 = f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + (6H\xi - \xi_{,\phi}\dot{\phi})\dot{\phi}^3 + \rho, \quad (7)$$

$$3FH^2 + 2F\dot{H} = \frac{1}{2}(FR - f) - \ddot{F} - 2H\dot{F} + \dot{\phi}^2(2\xi\ddot{\phi} + \xi_{,\phi}\dot{\phi}^2) - P, \quad (8)$$

$$f_{,X}\ddot{\phi} + \left(3Hf_{,X} + \dot{f}_{,X}\right)\dot{\phi} - f_{,\phi} + T^{(c)} = 0, \quad (9)$$

$$\dot{\rho} + 3H(\rho + P) = 0, \quad (10)$$

where a dot represents a derivative with respect to  $t$ , and

$$H = \dot{a}/a, \quad R = 6(2H^2 + \dot{H}), \quad (11)$$

$$T^{(c)} = -2\dot{\phi} \left[ \dot{\phi}(4\xi_{,\phi}\ddot{\phi} + \xi_{,\phi\phi}\dot{\phi}^2) - 6\xi \left\{ 2H\ddot{\phi} + (3H^2 + \dot{H})\dot{\phi} \right\} \right]. \quad (12)$$

We also obtain the following linearized equations for the perturbed metric (6) in the Fourier space<sup>1</sup>:

$$3H(\dot{\Phi} + H\Psi) + \frac{k^2}{a^2}\Phi + \frac{1}{2F} \left[ -\frac{1}{2}(f_{,\phi}\delta\phi + f_{,X}\delta X) + \frac{1}{2}\dot{\phi}^2(f_{,X\phi}\delta\phi + f_{,XX}\delta X + F_{,X}\delta R) + f_{,X}\dot{\phi}\delta\phi - 3H\delta\dot{F} \right. \\ \left. + \left( 3H^2 + 3\dot{H} - \frac{k^2}{a^2} \right) \delta F + 3\dot{F}(\dot{\Phi} + H\Psi) + (3H\dot{F} - f_{,X}\dot{\phi}^2)\Psi - \delta T_0^{0(c)} + \delta\rho \right] = 0, \quad (13)$$

$$f_{,X} \left[ \delta\ddot{\phi} + \left( 3H + \frac{\dot{f}_{,X}}{f_{,X}} \right) \delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi - \dot{\phi}(3\dot{\Phi} + \dot{\Psi}) \right] - 2f_{,\phi}\Psi + \frac{1}{a^3}(a^3\dot{\phi}\delta f_{,X})' - \delta f_{,\phi} = -\delta T^{(c)} - 2\Psi T^{(c)}, \quad (14)$$

$$\Phi = \Psi + \frac{\delta F}{F}, \quad (15)$$

$$\delta\dot{\rho} + 3H(\delta\rho + \delta P) = (\rho + P) \left( 3\dot{\Phi} - \frac{k^2}{a^2}v \right), \quad (16)$$

$$\frac{1}{a^3(\rho + P)} \frac{d}{dt} [a^3(\rho + P)v] = \Psi + \frac{\delta P}{\rho + P}, \quad (17)$$

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<sup>1</sup> We checked that  $\delta T_0^{0(c)}$  and  $\delta T^{(c)}$  in Eqs. (18) and (19) coincide with those given in Ref. [54]. However there are some differences in Eqs. (13) and (14) compared to Ref. [54]. We do not have the term  $(\delta F/F)T_0^{0(c)}$  in Eq. (13) and we have the term  $2\Psi T^{(c)}$  in Eq. (14).

where  $k$  is a comoving wavenumber, and

$$\delta T_0^{0(c)} = \dot{\phi}^2 \left[ 3(\xi_{,\phi}\dot{\phi} - 6H\xi)\dot{\delta\phi} - 2\xi\frac{k^2}{a^2}\delta\phi + \dot{\phi} \left\{ \xi_{,\phi}\dot{\delta\phi} + (\xi_{,\phi\phi}\dot{\phi} - 6H\xi_{,\phi})\delta\phi \right\} \right. \\ \left. + 6\xi\dot{\phi}(\dot{\Phi} + H\Psi) - 2\dot{\phi}(2\xi_{,\phi}\dot{\phi} - 9H\xi)\Psi \right], \quad (18)$$

$$\delta T^{(c)} = -2 \left\{ 4\dot{\phi}(\xi_{,\phi}\dot{\phi} - 3H\xi)\ddot{\delta\phi} + [4\dot{\phi}(\xi_{,\phi\phi}\dot{\phi}^2 + 2\xi_{,\phi}\ddot{\phi}) - 12\xi(H\ddot{\phi} + 3H^2\dot{\phi} + \dot{H}\dot{\phi})]\dot{\delta\phi} \right. \\ + \left[ \dot{\phi}^2(\xi_{,\phi\phi\phi}\dot{\phi}^2 + 4\xi_{,\phi\phi}\ddot{\phi}) - 6\xi_{,\phi}\dot{\phi}(2H\ddot{\phi} + 3H^2\dot{\phi} + \dot{H}\dot{\phi}) - 4\xi(\ddot{\phi} + 2H\dot{\phi})\frac{k^2}{a^2} \right] \delta\phi \\ + 6\xi\dot{\phi} \left[ 2(\dot{\Phi} + H\Psi)(\ddot{\phi} + 3H\dot{\phi}) + \dot{\phi}(\ddot{\Phi} + \dot{H}\Psi + H\dot{\Psi}) \right] - 4\dot{\phi}^2(\xi_{,\phi}\dot{\phi} - 3H\xi)\dot{\Psi} \\ \left. - 4\dot{\phi}^2(\xi_{,\phi\phi}\dot{\phi}^2 + 4\xi_{,\phi}\ddot{\phi})\Psi + 2\xi\dot{\phi} \left[ 9(2H\ddot{\phi} + 2H^2\dot{\phi} + \dot{H}\dot{\phi}) - \dot{\phi}\frac{k^2}{a^2} \right] \Psi \right\}. \quad (19)$$

We introduce the gauge-invariant matter density perturbation  $\delta_m$ , as

$$\delta_m \equiv \frac{\delta\rho}{\rho} + 3H(1+w)v, \quad (20)$$

where  $w \equiv P/\rho$ .

### III. QUASI-STATIC APPROXIMATION FOR PERTURBATIONS OF NON-RELATIVISTIC MATTER

The action (1) covers a wide variety of dark energy theories such as  $f(R)$  gravity [5], Brans-Dicke theory [37], scalar-tensor theory [7], and k-essence [55] (ghost condensate, tachyon, Dirac-Born-Infeld models, etc). Most of those theories take the Lagrangian of the form

$$f(R, \phi, X) = f_1(R, \phi) + f_2(\phi, X). \quad (21)$$

In this case the quantity

$$F = \frac{\partial f}{\partial R} = \frac{\partial f_1(R, \phi)}{\partial R}, \quad (22)$$

is a function of  $R$  and  $\phi$ . Note that the Lagrangian (21) does not include theories in which the kinetic term  $X$  couples to the Ricci scalar  $R$ .

In Ref. [41] the perturbation equation for non-relativistic matter has been derived for the Lagrangian (21) with  $\xi(\phi) = 0$ , under the assumption that the effective mass of the field  $\phi$  is smaller than the Hubble expansion rate  $H$ . In this work we shall take into account the effective mass  $M_\phi$  of the field by defining

$$M_\phi^2 \equiv -f_{,\phi\phi}/2. \quad (23)$$

For a minimally coupled scalar field with the Lagrangian  $f = R/(8\pi G) + 2X - 2V(\phi)$ , we have that  $M_\phi^2 = V_{,\phi\phi}$ . The quantity  $M_\phi$  does not in general correspond to the mass of the propagating canonical field.

For the perfect fluid let us consider non-relativistic matter ( $w = 0$ ). Combining Eqs. (16) and (17), the gauge-invariant matter perturbation  $\delta_m = \delta\rho/\rho + 3Hv$  satisfies

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi = 3\ddot{B} + 6H\dot{B}, \quad (24)$$

where  $B \equiv \Phi + Hv$ . The modes relevant to the observations of large-scale structure correspond to the sub-horizon perturbations ( $k \gg aH$ ). In order to derive the equation of matter perturbations approximately, we use the quasi-static approximation under which the dominant terms in Eqs. (13), (14), and (24) are those including  $k^2/a^2$ ,  $\delta\rho$  (or  $\delta_m$ ), and  $M_\phi^2$  [41, 42, 46, 52]. As long as the oscillating mode of a scalar-field degree of freedom is suppressed relative

to the mode induced by matter perturbations, this approximation is valid for sub-horizon perturbations [20, 22, 24]. First of all, the perturbation in the Ricci scalar is given by

$$\begin{aligned}\delta R &= 2 \left[ \left( \frac{k^2}{a^2} - 3\dot{H} \right) \Psi - 2 \frac{k^2}{a^2} \Phi - 3(\ddot{\Phi} + 4H\dot{\Phi} + H\dot{\Psi} + \dot{H}\Psi + 4H^2\Psi) \right] \\ &\simeq 2 \frac{k^2}{a^2} (\Psi - 2\Phi) .\end{aligned}\tag{25}$$

From Eqs. (15) and (25) it follows that

$$\delta R \simeq -\frac{2k^2}{a^2} \frac{\Psi + (2F_{,\phi}/F)\delta\phi}{1 + 4r_1},\tag{26}$$

where

$$r_1 \equiv \frac{k^2}{a^2} \frac{F_{,R}}{F} \simeq \frac{k^2}{3a^2} \frac{1}{M_R^2}.\tag{27}$$

Here  $M_R^2 \equiv F/(3F_{,R})$  is the mass squared of a scalar-field degree of freedom coming from the curvature term [14, 20] (which is valid in the region  $M_R^2 \gg H^2$ ).

For the theory (21) the perturbation  $\delta f$  is given by

$$\delta f = F\delta R + f_{,\phi}\delta\phi + f_{2,X}\delta X,\tag{28}$$

where  $\delta X = \dot{\phi}\delta\phi - \dot{\phi}^2\Psi$ . Under the quasi-static approximation we have

$$\delta f_{,\phi} \simeq F_{,\phi}\delta R - 2M_\phi^2\delta\phi.\tag{29}$$

From Eq. (14) we find

$$f_{,X} \frac{k^2}{a^2} \delta\phi - F_{,\phi}\delta R + 2M_\phi^2\delta\phi \simeq -\delta T^{(c)},\tag{30}$$

where

$$\delta T^{(c)} \simeq 4\xi \frac{k^2}{a^2} \left[ 2(\ddot{\phi} + 2H\dot{\phi})\delta\phi + \dot{\phi}^2\Psi \right].\tag{31}$$

Here we have neglected the oscillating mode of the field perturbation  $\delta\phi$ . If the field is sufficiently heavy in the regions of high density, such an oscillating mode can be important as we go back to the past [20, 22, 56]. The initial conditions for the field perturbation need to be chosen so that the oscillating mode is suppressed relative to the matter-induced mode.

From Eqs. (26) and (30), it follows that

$$\delta\phi \simeq -\frac{2F_{,\phi} + 4\xi\dot{\phi}^2(1 + 4r_1)}{(1 + 4r_1)[f_{,X} + 2r_2 + 8\xi(\ddot{\phi} + 2H\dot{\phi})] + 4F_{,\phi}^2/F} \Psi,\tag{32}$$

$$\delta R \simeq -\frac{2k^2}{a^2} \frac{f_{,X} + 2r_2 + 8\xi(\ddot{\phi} + 2H\dot{\phi} - \dot{\phi}^2 F_{,\phi}/F)}{(1 + 4r_1)[f_{,X} + 2r_2 + 8\xi(\ddot{\phi} + 2H\dot{\phi})] + 4F_{,\phi}^2/F} \Psi,\tag{33}$$

where

$$r_2 \equiv \frac{a^2}{k^2} M_\phi^2.\tag{34}$$

From Eq. (13) one has

$$\frac{k^2}{a^2} \Phi \simeq \frac{1}{2F} \left( \frac{k^2}{a^2} \delta F - \delta\rho - 2\xi\dot{\phi}^2 \frac{k^2}{a^2} \delta\phi \right),\tag{35}$$

which, together with Eq. (15), gives

$$\frac{k^2}{a^2} \Psi \simeq -\frac{1}{2F} \left( \frac{k^2}{a^2} \delta F + \delta\rho + 2\xi\dot{\phi}^2 \frac{k^2}{a^2} \delta\phi \right).\tag{36}$$

Plugging Eqs. (32) and (33) into Eq. (36), we obtain

$$\frac{k^2}{a^2} \Psi \simeq -\frac{\delta\rho}{2F} \frac{(1+4r_1)(f_{,X}+2r_2)+4F_{,\phi}^2/F+8\xi(1+4r_1)(\ddot{\phi}+2H\dot{\phi})}{(1+3r_1)(f_{,X}+2r_2)+3F_{,\phi}^2/F+2\xi[4(1+3r_1)(\ddot{\phi}+2H\dot{\phi})-2F_{,\phi}\dot{\phi}^2/F-2\xi\dot{\phi}^4(1+4r_1)/F]} , \quad (37)$$

$$\frac{k^2}{a^2} \Phi \simeq -\frac{\delta\rho}{2F} \frac{(1+2r_1)(f_{,X}+2r_2)+2F_{,\phi}^2/F+4\xi[2(1+2r_1)(\ddot{\phi}+2H\dot{\phi})-F_{,\phi}\dot{\phi}^2/F]}{(1+3r_1)(f_{,X}+2r_2)+3F_{,\phi}^2/F+2\xi[4(1+3r_1)(\ddot{\phi}+2H\dot{\phi})-2F_{,\phi}\dot{\phi}^2/F-2\xi\dot{\phi}^4(1+4r_1)/F]} . \quad (38)$$

Under the quasi-static approximation on sub-horizon scales the r.h.s. of Eq. (24) can be neglected relative to the l.h.s. of it. Moreover the gauge-invariant matter perturbation can be approximated as  $\delta_m \simeq \delta\rho/\rho$ . Then, from Eqs. (24) and (37), we finally obtain

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}} \rho \delta_m \simeq 0 . \quad (39)$$

Here the effective gravitational coupling is given by

$$G_{\text{eff}} = \frac{1}{8\pi F} \frac{1+4r_1}{1+3r_1} \left\{ 1 + \frac{[F_{,\phi} + 2(1+4r_1)\xi\dot{\phi}^2]^2}{(1+4r_1)\mu F} \right\} , \quad (40)$$

where

$$\mu \equiv (1+3r_1)(f_{,X}+2r_2)+3F_{,\phi}^2/F+2\xi[4(1+3r_1)(\ddot{\phi}+2H\dot{\phi})-2F_{,\phi}\dot{\phi}^2/F-2\xi\dot{\phi}^4(1+4r_1)/F] . \quad (41)$$

We define the following quantity

$$\zeta \equiv \Psi/\Phi . \quad (42)$$

From Eqs. (37) and (38) it follows that

$$\zeta \simeq \frac{(1+4r_1)(f_{,X}+2r_2)+4F_{,\phi}^2/F+8\xi(1+4r_1)(\ddot{\phi}+2H\dot{\phi})}{(1+2r_1)(f_{,X}+2r_2)+2F_{,\phi}^2/F+4\xi[2(1+2r_1)(\ddot{\phi}+2H\dot{\phi})-F_{,\phi}\dot{\phi}^2/F]} . \quad (43)$$

We also introduce the effective gravitational potential

$$\Phi_{\text{eff}} = (\Psi + \Phi)/2 . \quad (44)$$

This quantity characterizes the deviation of light rays, which is linked with the Integrated Sachs-Wolfe (ISW) effect in CMB and weak lensing observations [57]. From Eqs. (37) and (38) we have

$$\Phi_{\text{eff}} \simeq -\frac{\rho \delta_m a^2}{2F k^2} \frac{(1+3r_1)[f_{,X}+2r_2+8\xi(\ddot{\phi}+2H\dot{\phi})]+3F_{,\phi}^2/F-2\xi F_{,\phi}\dot{\phi}^2/F}{(1+3r_1)[f_{,X}+2r_2+8\xi(\ddot{\phi}+2H\dot{\phi})]+3F_{,\phi}^2/F-4\xi[F_{,\phi}\dot{\phi}^2/F+\xi\dot{\phi}^4(1+4r_1)/F]} . \quad (45)$$

Note that for the theories with  $\xi = 0$  this reduces to

$$\Phi_{\text{eff}} \simeq -\frac{\rho \delta_m a^2}{2F k^2} . \quad (46)$$

The presence of the  $\xi$  term leads to the modification to the effective gravitational potential and hence this can leave some nontrivial signature on the CMB spectrum [36, 49].

#### IV. EVOLUTION OF PERTURBATIONS IN SPECIFIC THEORIES

In this section we apply the results in the previous section to a number of concrete modified gravitational theories. It is convenient to write the matter perturbation equation (39) in the form

$$\delta_m'' + \left( \frac{1}{2} - \frac{3}{2} w_{\text{eff}} \right) \delta_m' - (12\pi F G_{\text{eff}}) \Omega_m \delta_m \simeq 0 , \quad (47)$$

where a prime represents a derivative with respect to  $N \equiv \ln a$ , and

$$w_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2} , \quad \Omega_m \equiv \frac{\rho}{3FH^2} . \quad (48)$$

### A. $f(R)$ gravity with $\xi = 0$

Let us consider the theories with the Lagrangian

$$f(R, \phi, X) = f_1(R) + f_2(\phi, X), \quad (49)$$

which includes  $f(R)$  gravity as a specific case. When  $\xi = 0$ , Eqs. (37), (38), (40), and (43) give

$$\frac{k^2}{a^2}\Psi \simeq -\frac{\delta\rho}{2F}\frac{1+4r_1}{1+3r_1}, \quad \frac{k^2}{a^2}\Phi \simeq -\frac{\delta\rho}{2F}\frac{1+2r_1}{1+3r_1}, \quad G_{\text{eff}} \simeq \frac{1}{8\pi F}\frac{1+4r_1}{1+3r_1}, \quad \zeta = \frac{1+4r_1}{1+2r_1}, \quad (50)$$

where  $\Phi_{\text{eff}}$  is given in Eq. (46). This matches with the results obtained in Refs. [41, 58] in the context of  $f(R)$  gravity. Even in the presence of a scalar field  $\phi$  minimally coupled to gravity, the above results are the same as those in  $f(R)$  gravity.

In the regime where the mass squared  $M_R^2$  of the scalar-field degree of freedom (scalon [59]) is much larger than  $k^2/a^2$ , i.e.  $r_1 \ll 1$ , the evolution of perturbations during the matter dominance ( $a \propto t^{2/3}$ ,  $w_{\text{eff}} \simeq 0$ , and  $\Omega_m \simeq 1$ ) is given by  $\delta_m \propto t^{2/3}$ ,  $\Phi_{\text{eff}} \simeq \text{constant}$ , and  $\zeta \simeq 1$ . In the regime  $M_R^2 \ll k^2/a^2$ , the perturbations during the matter era evolve as [20, 22]

$$\delta_m \propto t^{(\sqrt{33}-1)/6}, \quad \Phi_{\text{eff}} \propto t^{(\sqrt{33}-5)/6}, \quad \zeta \simeq 2. \quad (51)$$

After the perturbations enter the epoch of cosmic acceleration, the evolution (51) is subject to change.

### B. Scalar-tensor theories with $\xi = 0$

Let us consider scalar-tensor theories with  $\xi = 0$  in which the function  $f_1$  in the Lagrangian (21) depends on both  $R$  and  $\phi$ . This covers the theories with the effective two-scalar degrees of freedom, i.e. scalon (the gravitational scalar) and the field  $\phi$ . The scalar-tensor theory with  $f = F(\phi)R + f_2(\phi, X)$  has one scalar degree of freedom, but the nonlinear action in  $R$  gives rise to another gravitational scalar degree of freedom. From Eqs. (37), (38), (40), and (43), it follows that

$$\begin{aligned} \frac{k^2}{a^2}\Psi &\simeq -\frac{\delta\rho}{2F}\frac{(1+4r_1)(f_{,X}+2r_2)+4F_{,\phi}^2/F}{(1+3r_1)(f_{,X}+2r_2)+3F_{,\phi}^2/F}, & \frac{k^2}{a^2}\Phi &\simeq -\frac{\delta\rho}{2F}\frac{(1+2r_1)(f_{,X}+2r_2)+2F_{,\phi}^2/F}{(1+3r_1)(f_{,X}+2r_2)+3F_{,\phi}^2/F}, \\ G_{\text{eff}} &\simeq \frac{1}{8\pi F}\frac{1+4r_1+F_{,\phi}^2/(\mu F)}{1+3r_1}, & \zeta &\simeq \frac{(1+4r_1)(f_{,X}+2r_2)+4F_{,\phi}^2/F}{(1+2r_1)(f_{,X}+2r_2)+2F_{,\phi}^2/F}, \end{aligned} \quad (52)$$

where  $\mu = (1+3r_1)(f_{,X}+2r_2)+3F_{,\phi}^2/F$ . These correspond to the generalization of the results obtained in Ref. [41] with the mass term  $M_\phi$  taken into account ( $r_2 \neq 0$ ).

There are three different regimes depending on the values of  $r_1$  and  $r_2$ .

- (i)  $r_1 \ll 1$  and  $r_2 \gg F_{,\phi}^2/F$

This corresponds to the regime in which both the scalon and the field  $\phi$  are sufficiently heavy. The evolution of cosmological perturbations mimics that of GR.

- (ii)  $r_1 \ll 1$  and  $r_2 \ll F_{,\phi}^2/F$

In this regime the scalon is sufficiently heavy, whereas the field  $\phi$  is light such that it gives rise to the modification of gravity. In fact we have

$$\frac{k^2}{a^2}\Psi \simeq -\frac{\delta\rho}{2F}\frac{f_{,X}+4F_{,\phi}^2/F}{f_{,X}+3F_{,\phi}^2/F}, \quad \frac{k^2}{a^2}\Phi \simeq -\frac{\delta\rho}{2F}\frac{f_{,X}+2F_{,\phi}^2/F}{f_{,X}+3F_{,\phi}^2/F}, \quad G_{\text{eff}} \simeq \frac{1}{8\pi F}\frac{f_{,X}+4F_{,\phi}^2/F}{f_{,X}+3F_{,\phi}^2/F}, \quad \zeta = \frac{f_{,X}+4F_{,\phi}^2/F}{f_{,X}+2F_{,\phi}^2/F}. \quad (53)$$

Let us consider Brans-Dicke (BD) theory [37] with the field potential, i.e.

$$f = \frac{\phi}{\kappa}R + \frac{2\omega_{\text{BD}}}{\kappa\phi}X - 2V(\phi), \quad (54)$$



where  $\omega_{\text{BD}}$  is the BD parameter, and  $\kappa = 1/M_{\text{pl}}$  ( $M_{\text{pl}}$  is the reduced Planck mass). It then follows that

$$\frac{k^2}{a^2}\Psi \simeq -\frac{\delta\rho}{2F}\frac{2\omega_{\text{BD}}+4}{2\omega_{\text{BD}}+3}, \quad \frac{k^2}{a^2}\Phi \simeq -\frac{\delta\rho}{2F}\frac{2\omega_{\text{BD}}+2}{2\omega_{\text{BD}}+3}, \quad G_{\text{eff}} \simeq \frac{1}{8\pi F}\frac{2\omega_{\text{BD}}+4}{2\omega_{\text{BD}}+3}, \quad \zeta = \frac{\omega_{\text{BD}}+2}{\omega_{\text{BD}}+1}. \quad (55)$$

The no ghost condition in BD theory corresponds to  $\omega_{\text{BD}} > -3/2$ , in which case the gravitational coupling  $G_{\text{eff}}$  is positive. The evolution of perturbations during the matter-dominated epoch is given by

$$\delta_m \propto t^{\frac{1}{6}\left(\sqrt{\frac{99+50\omega_{\text{BD}}}{3+2\omega_{\text{BD}}}}-1\right)}, \quad \Phi_{\text{eff}} \propto t^{\frac{1}{6}\left(\sqrt{\frac{99+50\omega_{\text{BD}}}{3+2\omega_{\text{BD}}}}-5\right)}. \quad (56)$$

Compared to GR ( $\omega_{\text{BD}} \rightarrow \infty$ ), the growth rate of perturbations is enhanced for the theories with  $\omega_{\text{BD}} > -3/2$ . We have  $\zeta > 1$  for  $\omega_{\text{BD}} > -1$  and  $\zeta < -1$  for  $-3/2 < \omega_{\text{BD}} < -1$ .

- (iii)  $r_1 \gg 1$

This is the regime in which the scalaron has a light mass. Irrespective of the values of  $r_2$  one has

$$\frac{k^2}{a^2}\Psi \simeq -\frac{\delta\rho}{2F} \cdot \frac{4}{3}, \quad \frac{k^2}{a^2}\Phi \simeq -\frac{\delta\rho}{2F} \cdot \frac{2}{3}, \quad G_{\text{eff}} = \frac{1}{8\pi F} \cdot \frac{4}{3}, \quad \zeta = 2, \quad (57)$$

which correspond to the massless regime  $r_1 \gg 1$  in  $f(R)$  gravity.

We can consider models in which the sequence of three cosmological epochs (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii) occurs. If this occurs during the matter era, the matter perturbation starts to evolve as  $\delta_m \propto t^{2/3}$  in the region (i) and its evolution is followed by (56) [region (ii)] and (51) [region (iii)]. In this case the variable  $\zeta$  evolves as  $1 \rightarrow (\omega_{\text{BD}} + 2)/(\omega_{\text{BD}} + 1) \rightarrow 2$ . If the perturbations reach the regime  $r_1 > 1$  before entering the regime  $r_2 < F_{,\phi}^2/F$ , then there is no intermediate regime (ii) characterized by Eq. (53).

### C. Brans-Dicke theory with $\xi \neq 0$

Let us consider BD theory with a potential described by the Lagrangian (54) in the presence of the field self-interaction  $\xi(\phi)\Box\phi(\partial^\mu\phi\partial_\mu\phi)$ . From Eqs. (37), (38), (40), (43), and (45), it follows that

$$\frac{k^2}{a^2}\Psi \simeq -\frac{\kappa\delta\rho}{2\phi}\frac{2\omega_{\text{BD}}+4+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})}{2\omega_{\text{BD}}+3+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})-4\xi\kappa(\dot{\phi}^2+\xi\kappa\dot{\phi}^4)}, \quad (58)$$

$$\frac{k^2}{a^2}\Phi \simeq -\frac{\kappa\delta\rho}{2\phi}\frac{2\omega_{\text{BD}}+2+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})-4\xi\kappa\dot{\phi}^2}{2\omega_{\text{BD}}+3+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})-4\xi\kappa(\dot{\phi}^2+\xi\kappa\dot{\phi}^4)}, \quad (59)$$

$$G_{\text{eff}} \simeq \frac{\kappa}{8\pi\phi}\left[1+\frac{(1+2\xi\kappa\dot{\phi}^2)^2}{\kappa\phi\mu}\right], \quad (60)$$

$$\zeta \simeq \frac{2\omega_{\text{BD}}+4+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})}{2\omega_{\text{BD}}+2+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})-4\xi\kappa\dot{\phi}^2}, \quad (61)$$

$$\Phi_{\text{eff}} \simeq -\frac{\kappa\rho}{2\phi}\frac{a^2}{k^2}\delta_m\frac{2\omega_{\text{BD}}+3+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})-2\xi\kappa\dot{\phi}^2}{2\omega_{\text{BD}}+3+2r_2\kappa\phi+8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})-4\xi\kappa(\dot{\phi}^2+\xi\kappa\dot{\phi}^4)}, \quad (62)$$

where

$$\kappa\phi\mu = 2\omega_{\text{BD}}+3+2r_2\kappa\phi+2\xi\kappa[4\phi(\ddot{\phi}+2H\dot{\phi})-2\dot{\phi}^2-2\xi\kappa\dot{\phi}^4]. \quad (63)$$

For the special case with  $r_2 = 0$  and  $\xi(\phi) = 1/(M\phi^2)$  ( $M$  is a constant having a dimension of mass), the effective gravitational coupling (60) agrees with the one derived in Ref. [36].

In the presence of the field potential  $V(\phi)$ , one can recover the results in GR by taking the massive limit  $r_2\kappa\phi \rightarrow \infty$  in Eqs. (58)-(62). There is another mechanism called the Vainshtein mechanism [27] for the recovery of GR in the high-curvature regime (i.e. during radiation and deep matter eras). This is the case in which the term  $8\xi\kappa\phi(\ddot{\phi}+2H\dot{\phi})$  is the dominant contributions in Eqs. (58)-(62). In fact this situation arises for the function  $\xi(\phi)$  responsible for the late-time cosmic acceleration. For the choice  $\xi(\phi) = 1/(M\phi^2)$ , provided  $\omega_{\text{BD}} < -2$ , there is a stable de Sitter solution with  $\dot{H} = 0$  and  $x \equiv \dot{\phi}/(H\phi) = \text{constant}$  even in the absence of the field potential ( $r_2 = 0$ ) [36, 38].



For the choice  $\xi = 1/(M\dot{\phi}^2)$ , one has  $\xi\kappa\phi(\ddot{\phi} + 2H\dot{\phi}) \gg 1$  and  $x \equiv \dot{\phi}/(H\phi) \ll 1$  in the early cosmological epoch [36]. During the matter era, for example, we have that  $\xi\kappa H\dot{\phi} = \kappa H^2 x/M \simeq 1/(6x) \gg 1$ , where we have used  $y \equiv \kappa H^2 x^2/M \simeq 1/6$  [38]. In this regime the perturbation equations (58)-(62) recover the results in GR. At late times the deviation from GR arises, so that the evolution of perturbations is subject to change. As we will see later, the avoidance for the appearance of ghosts demands the condition  $x > 0$ . Under this condition the  $\kappa\phi\mu$  term defined in Eq. (63) can remain to be positive [36]. Hence the growth rate of matter perturbations gets larger than that in GR, unlike the DGP model [47]. We also note that the presence of the nonlinear self-interaction gives rise to a nontrivial contribution to  $\Phi_{\text{eff}}$  because the last fraction in Eq. (62) is different from 1.

The presence of the field potential  $V(\phi)$  (i.e.  $r_2 \neq 0$ ) gives rise to the change for the evolution of perturbations. One may consider theories in which the field is heavy ( $r_2\kappa\phi \gg 1$ ) in the early cosmological epoch, but the presence of the nonlinear self-interaction leads to the recovery of GR instead of the chameleon mechanism based on a heavy scalar field. More specifically this corresponds to the condition  $\xi\kappa\phi(\ddot{\phi} + 2H\dot{\phi}) \gg r_2\kappa\phi \gg 1$ . Depending on the evolution of the terms  $\xi\kappa\phi(\ddot{\phi} + 2H\dot{\phi})$  and  $r_2\kappa\phi$ , the perturbations evolve differently at late times.

For more general theories with the Lagrangian  $f = F(\phi)R + 2\omega(\phi)X - 2V(\phi)$  (i.e.  $r_1 = 0$ ) the cosmological Vainshtein mechanism mentioned above can be also at work, provided that the  $\xi(\ddot{\phi} + 2H\dot{\phi})$  term is the dominant contributions in Eqs. (37), (38), (40), (43), and (45).

## V. CONDITIONS FOR THE AVOIDANCE OF GHOSTS AND INSTABILITIES

In this section, we study the full Lagrangian perturbed at second order without using any approximation to derive conditions for the avoidance of ghosts and Laplacian instabilities. Let us consider the following general action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(R, \phi, X) + \xi(\phi) \square \phi (\partial^\mu \phi \partial_\mu \phi) + P(Z) \right], \quad Z = -\frac{1}{2} \partial^\mu \chi \partial_\mu \chi, \quad (64)$$

where  $P(Z)$  models the Lagrangian for the matter fluid  $\chi$  with the following parameterized equation of state

$$\rho = 2ZP'(Z) - P(Z), \quad P = P(Z). \quad (65)$$

Here a prime represents a derivative with respect to  $Z$ . The sound speed squared of this fluid is [60]

$$c_s^2 = \frac{P'(Z)}{2ZP''(Z) + P'(Z)}. \quad (66)$$

As long as scalar and tensor perturbations are concerned, the above k-essence description of the matter field  $\chi$  is equivalent to the description for the barotropic fluid.

For this action we generally have three propagating scalar degrees of freedom, one from gravity, one from the field  $\phi$ , and one from the matter fluid. The action (64) can be written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (R - \lambda) F + \frac{1}{2} f(\lambda, \phi, X) + \xi(\phi) \square \phi (\partial^\mu \phi \partial_\mu \phi) + P(Z) \right], \quad (67)$$

where the equations of motion for the auxiliary fields lead to

$$\lambda = R, \quad F = \frac{\partial f}{\partial \lambda}. \quad (68)$$

Variation of the action (67) leads to the same equations of motion as those derived by varying the action (64). Note that  $F$  in general depends not only on  $\lambda = R$  but also on  $\phi$  and  $X$ . One example of theories with  $F_{,X} \neq 0$  is  $f = M_{\text{pl}}^2 R + \alpha R^2 + X/(R^2/M_{\text{pl}}^4)^m - V(\phi)$  with  $m > 3/2$  [61].

Formally, the auxiliary field  $\lambda$  can be eliminated by solving its equation of motion  $F = \partial f / \partial \lambda$ . In practice, this procedure can be done at each order in perturbative expansion. At the linear order we have

$$\delta\lambda = \frac{\delta F - f_{,R\phi} \delta\phi - f_{,RX} \delta X}{f_{,RR}}, \quad (69)$$

provided that  $f_{,RR} \neq 0$ . Using this relation to eliminate  $\delta\lambda$  at the level of the action only makes sense when  $f_{,RR} \neq 0$ . In subsection V A, we shall consider theories with  $f_{,RR} \neq 0$ .

The case with  $f_{,RR} = 0$  has measure zero in the space of theories. Indeed, adding e.g.,  $\alpha R^2$  with an extremely small  $\alpha$  to  $f$  would make  $f_{,RR}$  non-vanishing. Therefore, physically speaking, it is not really necessary to study theories with strictly vanishing  $f_{,RR}$ . Nonetheless, in subsection VB we shall consider the case with  $f_{,RR} = f_{,RX} = 0$  as a simple example of theories with  $f_{,RR} = 0$ . The case with  $f_{,RR} = 0$  and  $f_{,RX} \neq 0$  is not only physically irrelevant (having measure zero in the space of theories) but also technically complicated. Thus, we shall not address this case in the present paper.

### A. Case (i): $f_{,RR} \neq 0$

We consider the flat FLRW background plus scalar-type perturbations  $\alpha$ ,  $\beta$ ,  $\mathcal{R}$ , and  $E$ :

$$ds^2 = -(1 + 2\alpha)dt^2 + 2a(t)^2\beta_{,i}dtdx^i + a(t)^2[(1 - 2\mathcal{R})\delta_{ij} + 2E_{,ij}]dx^i dx^j. \quad (70)$$

By using the Faddeev-Jackiw method [62], we can calculate the reduced quadratic action for gauge-invariant perturbations (see also Refs. [63]). Since the perturbed Hamiltonian and momentum constraints are of the form

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}^{(2)}}{\partial \alpha} = -H\Pi_{\mathcal{R}} + \dot{\phi}\Pi_{\delta\phi} + \dot{F}\Pi_{\delta F} + \dot{\chi}\Pi_{\delta\chi} + h_1(\mathcal{R}, \delta\phi, \delta F, \delta\chi, E), \\ 0 &= \frac{\partial \mathcal{H}^{(2)}}{\partial \beta} = \Pi_E + h_2(\mathcal{R}, \delta\phi, \delta F, \delta\chi, E), \end{aligned} \quad (71)$$

where  $\mathcal{H}^{(2)}$  is the quadratic Hamiltonian density and  $\Pi_{\mathcal{R}, \delta\phi, \delta F, \delta\chi, E}$  are momenta conjugate to  $\mathcal{R}, \delta\phi, \delta F, \delta\chi, E$  respectively, we can solve them with respect to  $\Pi_{\delta\chi}$  and  $\Pi_E$ .

Noting that the constraints are generators of gauge transformation, the form of the constraints enables us to identify the three gauge-invariant variables as

$$\mathcal{Q}_1 \equiv \frac{\mathcal{R}}{H} + \frac{\delta\chi}{\dot{\chi}}, \quad \mathcal{Q}_2 \equiv \frac{\delta\phi}{\dot{\phi}} - \frac{\delta\chi}{\dot{\chi}}, \quad \mathcal{Q}_3 \equiv \frac{\delta F}{\dot{F}} - \frac{\delta\chi}{\dot{\chi}}. \quad (72)$$

By substituting the solutions of the Hamiltonian and momentum constraints to the quadratic action

$$S^{(2)} = \int d^4x \left[ \Pi_{\mathcal{R}}\dot{\mathcal{R}} + \Pi_{\delta\phi}\delta\dot{\phi} + \Pi_{\delta F}\delta\dot{F} + \Pi_{\delta\chi}\delta\dot{\chi} + \Pi_E\delta\dot{E} - \mathcal{H}^{(2)} \right], \quad (73)$$

we obtain the reduced quadratic action written in terms of the gauge-invariant variables  $\mathcal{Q}_i$  ( $i = 1, 2, 3$ ) and their conjugate momenta  $\Pi_{\mathcal{Q}_i}$  only. After using the equations of motion for the conjugate momenta to express them in terms of  $\mathcal{Q}_i$  and  $\dot{\mathcal{Q}}_i$ , the reduced quadratic action is expressed in terms of  $\mathcal{Q}_i$  and  $\dot{\mathcal{Q}}_i$  as

$$S^{(2)} = \frac{1}{2} \int d^4x a^3 \left[ \dot{\vec{\mathcal{Q}}}^t \mathbf{K} \dot{\vec{\mathcal{Q}}} - \frac{1}{a^2} \nabla \vec{\mathcal{Q}}^t \mathbf{G} \nabla \vec{\mathcal{Q}} - \vec{\mathcal{Q}}^t \mathbf{B} \vec{\mathcal{Q}} - \vec{\mathcal{Q}}^t \mathbf{M} \vec{\mathcal{Q}} \right], \quad (74)$$

where  $\mathbf{K}$ ,  $\mathbf{G}$ ,  $\mathbf{B}$  and  $\mathbf{M}$  are  $3 \times 3$  matrices.

#### 1. No ghost conditions

The three eigenvalues  $K_i$  ( $i = 1, 2, 3$ ) of the kinetic term matrix  $\mathbf{K}$  are all positive if and only if the following three combinations are positive.

$$\begin{aligned} K_1 K_2 K_3 &= \alpha_1 B / D, \\ K_1 K_2 + K_2 K_3 + K_3 K_1 &= (\alpha_2 B + \alpha_3) / D, \\ K_1 + K_2 + K_3 &= (\alpha_4 B + \alpha_5) / D, \end{aligned} \quad (75)$$

where

$$\begin{aligned}
B &= 24\xi H\dot{\phi} - 8\xi_{,\phi}\dot{\phi}^2 + f_{,X} + f_{,XX}\dot{\phi}^2 - F_{,X}^2\dot{\phi}^2/F_{,R}, \\
D &= (\dot{F} + 2HF - 2\xi\dot{\phi}^3)^2, \\
\alpha_1 &= 3(\rho + P)c_s^{-2}H^2\dot{\phi}^2\dot{F}F, \\
\alpha_2 &= [(\dot{F}^2 + 4H^2F^2)(\rho + P)c_s^{-2} + 18H^2\dot{F}^2F]\dot{\phi}^2/2, \\
\alpha_3 &= 6(\dot{F}^2 + 4\xi^2\dot{\phi}^6)(\rho + P)c_s^{-2}H^2F, \\
\alpha_4 &= [(\dot{F} + HF)^2 + 3H^2F^2]\dot{\phi}^2, \\
\alpha_5 &= (\dot{F}^2 + 4H^2F^2 + 4\xi^2\dot{\phi}^6)(\rho + P)c_s^{-2} + 12[(\dot{F} - \xi\dot{\phi}^3)^2 + 3\xi^2\dot{\phi}^6]H^2F,
\end{aligned} \tag{76}$$

and  $F = f_{,R}$ .

We assume that  $F > 0$  since this is required for the absence of tensor ghosts. Let us also suppose that  $(\rho + P)c_s^{-2} = 2(2ZZP'' + P')Z > 0$ , which is required for the absence of ghosts in the matter sector. Under these assumptions,  $D$  and  $\alpha_i$  ( $i = 1, \dots, 5$ ) are non-negative, and there are no ghosts in the scalar sector if and only if  $B > 0$ , i.e.

$$24\xi H\dot{\phi} - 8\xi_{,\phi}\dot{\phi}^2 + f_{,X} + f_{,XX}\dot{\phi}^2 - F_{,X}^2\dot{\phi}^2/F_{,R} > 0. \tag{77}$$

## 2. Avoidance of Laplacian instabilities

For large  $k$  the dispersion relation is specified by

$$\det(\omega^2 \mathbf{K} - k^2 \mathbf{G}/a^2) = 0. \tag{78}$$

This is equivalent to

$$\det(c^2 \mathbf{K} - \mathbf{G}) = 0, \tag{79}$$

where  $c$  is the speed of propagation defined by

$$\omega^2 = c^2 \frac{k^2}{a^2}. \tag{80}$$

More explicitly we have

$$\det(c^2 \mathbf{K} - \mathbf{G}) = 3(\rho + P)BH^2\dot{\phi}^2\dot{F}F(c^2 - c_s^2) [c^4 - (c_1^2 + c_2^2)c^2 + c_1^2c_2^2] / D, \tag{81}$$

where

$$c_1^2 + c_2^2 = 1 + \alpha_6/B, \quad c_1^2c_2^2 = (\alpha_6 - \alpha_7)/B, \tag{82}$$

and

$$\alpha_6 = f_{,X} + 8(\ddot{\phi} + 2H\dot{\phi})\xi, \quad \alpha_7 = 16\xi^2\dot{\phi}^4/(3F). \tag{83}$$

Gradient instabilities are avoided if and only if  $c_s^2$ ,  $c_1^2 + c_2^2$ ,  $c_1^2c_2^2$  and  $(c_1^2 - c_2^2)^2$  are all non-negative, where

$$(c_1^2 - c_2^2)^2 = (1 - \alpha_6/B)^2 + 4\alpha_7/B. \tag{84}$$

Under the conditions  $F > 0$  (absence of tensor ghosts) and  $B > 0$  (absence of scalar ghosts), the condition for the absence of gradient instabilities is equivalent to  $\alpha_6 > \alpha_7$ , i.e.

$$f_{,X} + 8(\ddot{\phi} + 2H\dot{\phi})\xi - 16\xi^2\dot{\phi}^4/(3F) > 0. \tag{85}$$

### 3. Mass of propagating modes

We provide here a straightforward procedure on how to define the mass for the propagating modes. For the reduced action (74), we can first diagonalize and normalize the kinetic matrix  $\mathbf{K}$  to the identity matrix, through a convenient linear field redefinition  $\vec{Q} = \mathbf{A} \vec{Q}'$ . Each field redefinition, in the  $k \rightarrow 0$  limit, will contribute to the matrices  $\mathbf{B}$  and  $\mathbf{M}$ , as  $\mathbf{A}$  is time-dependent and the modes feel the time evolution of the background. Having the canonically normalized kinetic matrix, we can perform a rotation  $\vec{Q}' = \mathbf{R} \vec{Q}''$ , which transforms the mass matrix into

$$\mathbf{M}'' = \mathbf{M}' + \mathbf{T}^2 - (\mathbf{B}'\mathbf{T} + \mathbf{T}\mathbf{B}')/2, \quad (86)$$

where  $\mathbf{T} = \dot{\mathbf{R}}^t \mathbf{R}$  is antisymmetric. Then the mass eigenvalues are obtained by choosing the elements of  $\mathbf{T}$  which diagonalize  $\mathbf{M}''$ . This procedure implies solving a differential equation for the elements of  $\mathbf{T}$  and  $\dot{\mathbf{T}}$ . We shall not present explicit expressions as they are rather complicated.

### 4. Specific theories

Let us apply the above results to the relaxation mechanism of the cosmological constant [61]. In the simplest case the Lagrangian is specified as

$$f = M_{\text{pl}}^2 R + \alpha R^2 + \frac{X}{(R^2/M_{\text{pl}}^4)^m} - V(\phi), \quad \xi = 0, \quad (87)$$

with  $m > 3/2$ . It is straightforward to show that

$$B = f_{,X} - \frac{F_{,X}^2 \dot{\phi}^2}{F_{,R}} = \frac{1}{(R^2/M_{\text{pl}}^4)^m} \left[ 1 - \frac{4m^2 (R^2/M_{\text{pl}}^4)^{m-1}}{F_{,R}} \frac{\pi^2}{M_{\text{pl}}^4} \right], \quad (88)$$

where

$$\pi = \frac{\dot{\phi}}{(R^2/M_{\text{pl}}^4)^m}, \quad (89)$$

is the momentum conjugate to  $\phi$  and

$$F_{,R} = 2\alpha + m(2m+1)(R^2/M_{\text{pl}}^4)^{m-1} \frac{\pi^2}{M_{\text{pl}}^4}. \quad (90)$$

The attractor behavior

$$\pi^2 \sim \left( \frac{M_{\text{pl}}^3}{H} \right)^2, \quad (91)$$

and  $R \sim H^2$  imply that

$$(R^2/M_{\text{pl}}^4)^{m-1} \frac{\pi^2}{M_{\text{pl}}^4} \sim (H/M_{\text{pl}})^{2(2m-3)} \rightarrow 0 \quad (\text{as } H \rightarrow 0), \quad (92)$$

under the assumption  $m > 3/2$ . Therefore, at low energy, we have

$$F_{,R} \sim 2\alpha, \quad B \sim \frac{1}{(R^2/M_{\text{pl}}^4)^m} > 0. \quad (93)$$

This shows the absence of ghost. Gradient instability is also absent as

$$\alpha_6 = f_{,X} = \frac{1}{(R^2/M_{\text{pl}}^4)^m} > 0, \quad \alpha_7 = 0. \quad (94)$$

Indeed, the two speeds of propagation  $c_1$  and  $c_2$  are both unity in the low-energy regime.

### B. Case (ii): $f_{,RR} = f_{,RX} = 0$

In this subsection we consider the case with  $f_{,RR} = f_{,RX} = 0$  as a simple example of theories with  $f_{,RR} = 0$ . As already stated, theories with  $f_{,RR} = 0$  have measure zero in the space of theories in the sense that they can easily be transformed to theories with  $f_{,RR} \neq 0$  by inclusion of an additional term like  $\alpha R^2$  with small  $\alpha$  to  $f$ .

The theories with  $f_{,RR} = f_{,RX} = 0$  correspond to  $f = F(\phi)R + f_2(X, \phi)$ . One example of the theories with  $f_{,RR} = f_{,RX} = 0$  and  $f_{,XX} \neq 0$  is  $f = F(\phi)R - 2X + 2X^2/M^4$ . Since  $\delta F = F_{,\phi} \delta\phi$ , the two fields  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are sufficient to study the perturbed action. Although this fact simplifies the problem, the nonminimal coupling to  $R$  directly affects the propagation properties of the field  $\delta\phi$ .

Since  $\delta F = F_{,\phi} \delta\phi$ , we consider the original action (64) instead of the equivalent action (67). Specializing to the present case, we have

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R + \frac{1}{2} f_2(\phi, X) + \xi(\phi) \square \phi (\partial^\mu \phi \partial_\mu \phi) + P(Z) \right]. \quad (95)$$

By using the Faddeev-Jackiw method [62], we can calculate the reduced quadratic action for two gauge-invariant variables. The perturbed Hamiltonian and momentum constraints are of the form

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}^{(2)}}{\partial \alpha} = -H \Pi_{\mathcal{R}} + \dot{\phi} \Pi_{\delta\phi} + \dot{\chi} \Pi_{\delta\chi} + h_1(\mathcal{R}, \delta\phi, \delta\chi, E), \\ 0 &= \frac{\partial \mathcal{H}^{(2)}}{\partial \beta} = \Pi_E + h_2(\mathcal{R}, \delta\phi, \delta\chi, E). \end{aligned} \quad (96)$$

As in the previous case, we can solve them with respect to  $\Pi_{\delta\chi}$  and  $\Pi_E$ . The form of the constraints enables us to identify the two gauge-invariant variables as

$$\mathcal{Q}_1 \equiv \frac{\mathcal{R}}{H} + \frac{\delta\chi}{\dot{\chi}}, \quad \mathcal{Q}_2 \equiv \frac{\delta\phi}{\dot{\phi}} - \frac{\delta\chi}{\dot{\chi}}. \quad (97)$$

By substituting the solutions of the Hamiltonian and momentum constraints to the quadratic action and expressing the conjugate momenta in terms of  $\mathcal{Q}_i$  and  $\dot{\mathcal{Q}}_i$ , we obtain the reduced quadratic action of the form,

$$S^{(2)} = \frac{1}{2} \int d^4x a^3 \left[ \dot{\vec{\mathcal{Q}}}^t \mathbf{K} \dot{\vec{\mathcal{Q}}} - \frac{1}{a^2} \nabla \vec{\mathcal{Q}}^t \mathbf{G} \nabla \vec{\mathcal{Q}} - \vec{\mathcal{Q}}^t \mathbf{B} \dot{\vec{\mathcal{Q}}} - \vec{\mathcal{Q}}^t \mathbf{M} \vec{\mathcal{Q}} \right], \quad (98)$$

where  $\mathbf{K}$ ,  $\mathbf{G}$ ,  $\mathbf{B}$  and  $\mathbf{M}$  are now  $2 \times 2$  matrices.

#### 1. No ghost conditions

The two eigenvalues  $K_i$  ( $i = 1, 2$ ) of the kinetic term matrix  $\mathbf{K}$  are all positive if and only if the following two combinations are positive:

$$\begin{aligned} K_1 K_2 &= \tilde{\alpha}_1 \tilde{B} / \tilde{D}, \\ K_1 + K_2 &= (\tilde{\alpha}_2 \tilde{B} + \tilde{\alpha}_3) / \tilde{D}, \end{aligned} \quad (99)$$

where

$$\begin{aligned} \tilde{B} &= (24\xi H \dot{\phi} - 8\xi_{,\phi} \dot{\phi}^2 + f_{,X} + f_{,XX} \dot{\phi}^2) F \dot{\phi}^2 + 3(\dot{F} - 2\xi \dot{\phi}^3)^2, \\ \tilde{D} &= (\dot{F} + 2HF - 2\xi \dot{\phi}^3)^2, \\ \tilde{\alpha}_1 &= 2(\rho + P) c_s^{-2} H^2 F, \\ \tilde{\alpha}_2 &= 4H^2 F \\ \tilde{\alpha}_3 &= [4F^2 H^2 + (\dot{F} - 2\xi \dot{\phi}^3)^2] (\rho + P) c_s^{-2}. \end{aligned} \quad (100)$$

The absence of ghosts for tensor perturbations demands the condition  $F > 0$ . The absence of ghosts in the matter sector requires that  $(\rho + P) c_s^{-2} = 2(2ZP'' + P')Z > 0$ . Under these assumptions,  $\tilde{D}$  and  $\tilde{\alpha}_i$  ( $i = 1, 2, 3$ ) are non-negative, and there are no ghosts in the scalar sector if and only if  $\tilde{B} > 0$ , i.e.

$$(24\xi H \dot{\phi} - 8\xi_{,\phi} \dot{\phi}^2 + f_{,X} + f_{,XX} \dot{\phi}^2) F \dot{\phi}^2 + 3(\dot{F} - 2\xi \dot{\phi}^3)^2 > 0. \quad (101)$$

## 2. Avoidance of Laplacian instabilities

For large  $k$  the dispersion relation is given by

$$\det(c^2 \mathbf{K} - \mathbf{G}) = 0, \quad (102)$$

where  $c$  is the speed of propagation defined by  $\omega^2 = c^2 k^2 / a^2$ .

There are two solutions to this equation: one is  $c^2 = c_s^2$  and another is

$$c^2 = \frac{[f_{2,X} + 8\xi(\ddot{\phi} + 2H\dot{\phi})]F\dot{\phi}^2 + (3\dot{F} + 2\xi\dot{\phi}^3)(\dot{F} - 2\xi\dot{\phi}^3)}{(24\xi H\dot{\phi} - 8\xi_{,\phi}\dot{\phi}^2 + f_{,X} + f_{,XX}\dot{\phi}^2)F\dot{\phi}^2 + 3(\dot{F} - 2\xi\dot{\phi}^3)^2}. \quad (103)$$

Under the no ghost condition (101), the condition for the absence of gradient instabilities corresponds to

$$[f_{2,X} + 8\xi(\ddot{\phi} + 2H\dot{\phi})]F\dot{\phi}^2 + (3\dot{F} + 2\xi\dot{\phi}^3)(\dot{F} - 2\xi\dot{\phi}^3) > 0. \quad (104)$$

## 3. Specific theories

Let us apply the above results to generalized Brans-Dicke theories recently proposed in Ref. [38]. These theories are given by

$$F(\phi) = \kappa^{1-n}\phi^{3-n}, \quad f_2(\phi, X) = 2\omega(\kappa\phi)^{1-n}X, \quad \xi(\phi) = M^{n-3}\phi^{-n}, \quad (105)$$

which can be obtained by demanding the existence of de Sitter solutions responsible for dark energy. The Brans-Dicke theory with  $\xi(\phi) = 1/(M\phi^2)$  [36] corresponds to the choice  $n = 2$ . The viable model parameter space is restricted in the regions  $2 \leq n \leq 3$  and  $\omega < -n(n-3)^2$  [38].

For this theory the no ghost condition (101) reduces to

$$\tilde{B} = (FH)^2 x [24y + x \{2\omega + 8ny + 3(3-n-2y)^2\}] > 0, \quad (106)$$

where  $x \equiv \dot{\phi}/(H\phi)$  and  $y \equiv x^2 M^{n-3} H^2 / \kappa^{1-n} > 0$ . During the radiation and deep matter eras one has  $x \ll y$  for the theories with  $2 \leq n < 3$ . Since  $\tilde{B} \simeq 24(FH)^2 xy$  in this regime, we require that  $x > 0$  to avoid the appearance of ghosts. The sign change of  $x$  leads to the violation of the condition (106) and hence the condition  $x > 0$  is required during the cosmological evolution from the radiation era to the de Sitter epoch [38].

Let us consider the evolution of the propagation speed (103) during the radiation and matter eras for  $2 \leq n < 3$ . In this regime the dominant terms in the numerator and the denominator of Eq. (103) correspond to  $8\xi(\ddot{\phi} + 2H\dot{\phi})F\dot{\phi}^2$  and  $24\xi HF\dot{\phi}^3$ , respectively. This gives

$$c^2 \simeq \frac{2}{3} + \frac{\ddot{\phi}}{3H\dot{\phi}}. \quad (107)$$

From Eqs. (8) and (9) we also obtain the following approximate relation

$$4H\xi\dot{\phi}\ddot{\phi} + 2\xi(3H^2 + \dot{H})\dot{\phi}^2 - F_{,\phi}(2H^2 + \dot{H}) \simeq 0. \quad (108)$$

As long as the field energy density is suppressed relative to the radiation and matter densities we have  $\dot{H}/H^2 \simeq -2 + \Omega_m/2$ , where  $\Omega_m \equiv \rho_m/(3FH^2)$  is the density parameter of matter. Plugging Eq. (108) into Eq. (107), it follows that

$$c^2 \simeq \frac{1}{12}(6 - \Omega_m) + \frac{3-n}{24y}\Omega_m. \quad (109)$$

Since  $y \simeq (3-n)\Omega_m/8$  during the radiation era [38], one has  $c^2 \simeq 5/6 - \Omega_m/12$ . Meanwhile  $y \simeq (3-n)/6$  during the matter era ( $\Omega_m \simeq 1$ ), so that  $c^2 \simeq 2/3$ . These results agree with those obtained in Refs. [36, 38]. At the de Sitter point one has  $0 \leq c^2 < 1$  as long as  $x > 0$  and  $n \geq 2$  [38].

## VI. CONCLUSIONS

We have discussed cosmological perturbations for the general Lagrangian  $f(R, \phi, X)/2 + \xi(\phi)\Box\phi(\partial^\mu\phi\partial_\mu\phi)$ , which includes most of modified gravity theories proposed in literature. The presence of the nonlinear field self-interaction term allows a possibility for the recovery of GR in the early cosmological epoch, even if the field potential is absent. We also take into account the effect of the field potential to cover the case in which the transition from the “massive” (GR) regime to the “massless” (scalar-tensor) regime occurs during the cosmological evolution by today.

Under the quasi-static approximation on sub-horizon scales we have derived the equations for the matter perturbation  $\delta_m$  and gravitational potentials  $\Psi$  and  $\Phi$  for the theories  $f = f_1(R, \phi) + f_2(\phi, X)$ , see Eqs. (37)-(41). There are two important mass scales given by  $M_R^2 = f_{,R}/(3f_{,RR})$  and  $M_\phi^2 = -f_{,\phi\phi}/2$ , which come from the gravitational scalar degree of freedom and the field  $\phi$  respectively. As long as both scalars are massive such that  $\{M_R^2, M_\phi^2\} \gg k^2/a^2$ , the perturbations are in the GR regime characterized by  $\delta_m \propto t^{2/3}$  and  $\Phi_{\text{eff}} = (\Psi + \Phi)/2 = \text{constant}$  during the matter-dominated epoch. The evolution of perturbations is subject to change in the regime  $M_R^2 \lesssim k^2/a^2$  or  $M_\phi^2 \lesssim k^2/a^2$ . The epoch at which the transition to the massless regime occurs depends on the models of dark energy.

In Sec. IV we have applied the results derived under the quasi-static approximation to a number of modified gravity theories and estimated the growth rate of perturbations. Not only with massive fields satisfying the conditions  $\{M_R^2, M_\phi^2\} \gg k^2/a^2$  but also in the presence of the nonlinear field self-interaction, the evolution of perturbations mimics that in GR during the early cosmological evolution. We showed that our results recover those obtained in Brans-Dicke theory with  $V(\phi) = 0$  and  $\xi(\phi) = 1/(M\phi^2)$  [36, 49] as a special case.

In Sec. V we have derived the conditions for the avoidance of ghosts and Laplacian instabilities by deriving the full second-order action without using any approximation. We have discussed two classes of theories: (i)  $f_{,RR} \neq 0$  and (ii)  $f_{,RR} = f_{,RX} = 0$ . The case (i) corresponds to theories with three scalar-field degrees of freedom, whereas in the case (ii) two scalar-field degrees of freedom are present. In the case (i) the conditions for the avoidance of ghosts and Laplacian instabilities are given by Eqs. (77) and (85) respectively, whereas in the case (ii) they are given by Eqs. (101) and (104) respectively. We have also shown that in generalized Brans-Dicke theories described by the functions (105) our results reproduce the no ghost condition as well as the propagation speed derived in Ref. [38].

It will be of interest to apply our results to the construction of viable modified gravity models of dark energy. In particular the genericity of the cosmological Vainshtein mechanism due to the scalar-field self-interaction should be explored further, together with local gravity constraints. The evolution of matter perturbations as well as gravitational potentials can be used to place constraints on modified gravity models from the observations of large-scale structure, CMB, and weak lensing. We hope that future observations and experiments will provide some distinguished features for the modification of gravity from General Relativity.

## ACKNOWLEDGEMENTS

The work of A. D and S. T. was supported by the Grant-in-Aid for Scientific Research Fund of the JSPS Nos. 09314 and 30318802. S. T. also thanks financial support for the Grant-in-Aid for Scientific Research on Innovative Areas (No. 21111006). The work of S. M. is supported by JSPS Grant-in-Aid for Young Scientists (B) No. 17740134, JSPS Grant-in-Aid for Creative Scientific Research No. 19GS0219, MEXT Grant-in-Aid for Scientific Research on Innovative Areas No. 21111006, JSPS Grant-in-Aid for Scientific Research (C) No. 21540278, the Mitsubishi Foundation, and World Premier International Research Center Initiative.

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